Probabilistic Methods in Combinatorics

Solutions to Assignment 9

Problem 1. Let $x_1, \dots, x_m$ be boolean variables. In mathematical logic, a *literal* is an atomic formula x_i or its negation $\neg x_i$. A *formula* φ is a *k-CNF* (conjunctive normal form) if it is a conjunction of *clauses* $C_1, C_2, \dots, C_n$ (i.e. $\varphi = C_1 \wedge C_2 \wedge \dots \wedge C_n$) where each clause C_i is a disjunction of k literals $y_{i,1}, \dots, y_{i,k}$ (i.e. $C_i = y_{i,1} \vee y_{i,2} \vee \dots \vee y_{i,k}$) corresponding to different variables. The formula φ is said to be *satisfiable* if it can be made true by assigning appropriate logical values (i.e., true/false) to its variables. This means that, under such assignment, for every clause in φ at least one of its literals is true. For example, the 2-CNF

$$(x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_1 \vee \neg x_2)$$

is satisfiable because for the assignment $(x_1, x_2) \mapsto (\text{true}, \text{false})$ we get

$$(\text{true} \vee \text{false}) \wedge (\text{false} \vee \text{true}) \wedge (\text{true} \vee \text{true}) = \text{true}$$

whereas the 2-CNF

$$(x_1 \vee x_2) \wedge (\neg x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (\neg x_1 \vee x_3) \wedge (\neg x_2 \vee \neg x_3)$$

is easily seen not to be satisfiable. The problem of deciding whether a k-CNF is satisfiable is called the k-SAT problem. For $k \geq 3$ this problem is known to be NP-complete. Show that any k-CNF in which every variable appears in at most $2^{k-2}/k$ clauses is satisfiable.

Solution. Let φ be a k-CNF in which every variable appears in at most $2^{k-2}/k$ clauses. Write $\varphi = C_1 \wedge C_2 \wedge \dots \wedge C_n$ where $C_i = y_{i,1} \vee y_{i,2} \vee \dots \vee y_{i,k}$. Let each boolean variable be true and false with probability $1/2$, independently of each other. For $1 \leq i \leq n$, let B_i be the event that clause C_i is false. Note that C_i is false if and only if $y_{i,1}, \dots, y_{i,k}$ are all false, which happens with probability $1/2^k$. Since each boolean variable appears in at most $2^{k-2}/k$ clauses, there are at most 2^{k-2} clauses which share a variable with any fixed C_i. One of those is C_i itself, so B_i is mutually independent of all but at most $2^{k-2} - 1$ other events B_j. Since $e \cdot \mathbb{P}(B_i) \cdot 2^{k-2} < 1$, the Lovász Local lemma implies that $\mathbb{P}(\cap_{i=1}^n \bar{B}_i) > 0$. This means that with positive probability all C_i are true, which means that φ is also true.

Problem 2. Let H be a d-uniform d-regular hypergraph (i.e. each edge consists of d vertices and each vertex is in precisely d edges). Show that if $d \geq 6$ then the vertices of H can be coloured with two colours, red and blue, such that every edge is nearly balanced, meaning that the number of red and blue vertices in every edge differ by at most $\sqrt{6d \log d}$.

Solution. Colour each vertex independently at random. For every edge $e \in E(H)$, let B_e be the event that the number of red and blue vertices on e differ by more than $\sqrt{6d \log d}$. By Chernoff's bound,

$$p = \mathbb{P}(B_e) \leq 2 \exp\left(-\frac{2(\sqrt{6d \log d}/2)^2}{d}\right) = 2\exp(-3 \log d) = \frac{2}{d^3}.$$

On the other hand, B_e is mutually independent from the set of those events B_f for which $e \cap f = \emptyset$. The number of edges $f \in E(H)$ with $e \cap f \neq \emptyset$ is at most d^2 since each vertex $v \in e$ is in d edges of H. So B_e is mutually independent from all but at most $d^2 - 1$ other events B_f. But $epd^2 \leq \frac{2e}{d} < 1$, so by Lovász Local Lemma, we have $\mathbb{P}(\bigcap_{e \in E(H)} \bar{B}_e) > 0$. This means that there exists a suitable colouring.

Problem 3. Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colours of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbours u of v such that c lies in $S(u)$. Prove that there is a proper colouring of G assigning to each vertex v a colour from its class $S(v)$.

Solution. We may assume that each $S(v)$ has size exactly $10d$ (otherwise, just discard elements of $S(v)$). Now colour each v independently at random with a colour in $S(v)$. For any two neighbouring vertices $u, v \in V(G)$ and colour $c \in S(u) \cap S(v)$, let $B_{u,v,c}$ be the event that both u and v are coloured c. Then $\mathbb{P}(B_{u,v,c}) = \frac{1}{(10d)^2}$. Note that $B_{u,v,c}$ is mutually independent of the set of events $B_{u',v',c'}$ with $\{u', v'\} \cap \{u, v\} = \emptyset$. How many events $B_{u',v',c'}$ are there with $\{u', v'\} \cap \{u, v\} \neq \emptyset$? Note that there are 2 ways to choose a common member of $\{u', v'\}$ and $\{u, v\}$ (say, without loss of generality, that $v' \in \{u, v\}$). Given a choice for that, there are $10d$ choices for c' (since $c' \in S(v')$). Finally, there are at most d choices for u' since u' must be a neighbour of v' with $c' \in S(u')$. Altogether, we find that there are at most $20d^2$ events $B_{u',v',c'}$ for which $\{u', v'\} \cap \{u, v\} \neq \emptyset$. So $B_{u,v,c}$ is mutually independent of all but at most $20d^2$ bad events.

Since $p = \mathbb{P}(B_{u,v,c}) = \frac{1}{100d^2}$ satisfies $ep(20d^2 + 1) < 1$, the Lovász Local Lemma implies that $\mathbb{P}\left(\bigcap_{u,v,c} \overline{B_{u,v,c}}\right) > 0$. This means that there exists a suitable proper colouring.

Remark. It would have been natural to consider for any edge $uv \in E(G)$ the bad event $B_{u,v}$ that u and v get the same colour. However, in this case we can only say $\mathbb{P}(B_{u,v}) \leq \frac{1}{10d}$ but we have possibly more than d^2 events $B_{u',v'}$ which are not independent from $B_{u,v}$ (e.g. all events of the form $B_{u,w}$ with $S(u) \cap S(w) \neq \emptyset$). So the Local lemma wouldn't apply in this case.